A refresher on probability theory

Statistical Natural Language Processing

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Why probability theory?

> *But it must be recognized that the notion 'probability of a sentence' is an entirely useless one, under any known interpretation of this term.* — Chomsky (1968)

Short answer: practice proved otherwise.

Slightly long answer

- Many linguistic phenomena are better explained as tendencies, rather than fixed rules
- Probability theory captures many characteristics of (human) cognition, language is not an exception

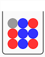
What is probability?

- Probability is a measure of (un)certainty
- We quantify the probability of an event with a number between 0 and 1
  - 0 the event is impossible
  - 0.5 the event is as likely to happen as it is not
  - 1 the event is certain
- The set of all possible *outcomes* of a trial is called *sample space* (Ω)
- An *event* (E) is a set of outcomes

Axioms of probability state that

1. $P(E) \in \mathbb{R}$, $P(E) \geqslant 0$
2. $P(\Omega) = 1$
3. For *disjoint* events E_1 and E_2, $P(E_1 \cup E_2) = P(E_1) + P(E_2)$

What you should already know



- P({🔵}) = 4/9
- P({🔴}) = 4/9
- P({⚪}) = 1/9
- P({🔵, 🔴}) = 8/9
- P({🔵, 🔴, ⚪}) = 1
- P({🔵🔵}) = 16/81
- P({🔴🔵}) = 16/81
- P({⚪🔵}) = 4/81
- P({⚪⚪}) = 1/81
- P({🔵🔴, ⚪🔴}) = 20/81

Where do probabilities come from

Axioms of probability do not specify how to assign probabilities to events.

Two major (rival) ways of assigning probabilities to events are

- Frequentist (objective) probabilities: probability of an event is its relative frequency (in the limit)
- Bayesian (subjective) probabilities: probabilities are degrees of belief

Random variables

- A random variable is a variable whose value is subject to uncertainties
- A random variable is always a number
- Think of a random variable as mapping between the outcomes of a trial to (a vector of) real numbers (a real valued function on the sample space)
- Example outcomes of uncertain experiments
  - height or weight of a person
  - length of a word randomly chosen from a corpus
  - whether an email is spam or not
  - the first word of a book, or first word uttered by a baby

  Note: not all of these are numbers

Random variables

mapping outcomes to real numbers

- Continuous
  - frequency of a sound signal: 100.5, 220.3, 4321.3 …
- Discrete
  - Number of words in a sentence: 2, 5, 10, …
  - Whether a review is negative or positive:

Outcome	Negative	Positive
Value	0	1

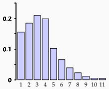
  - The POS tag of a word:

Outcome	Noun	Verb	Adj	Adv	…
Value	1	2	3	4	…
…or	10000	01000	00100	00010	…

Probability mass function

Example: probabilities for sentence length in words

- *Probability mass function* (*PMF*) of a *discrete* random variable (X) maps every possible (x) value to its probability ($P(X = x)$).



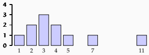
x	$P(X = x)$
1	0.155
2	0.185
3	0.210
4	0.194
5	0.102
6	0.066
7	0.039
8	0.023
9	0.012
10	0.005
11	0.004

Populations, distributions, samples

- A probability distribution characterizes a random variable
- We can define a distribution with a vector or table of probabilities, if we have a finite sample space
- Otherwise, we use (parametric) functions to map the (infinite) set of outcomes to probabilities
- Probability distributions characterize possibly infinite *populations*
- In most cases we have to work with *samples*

A sample from the distribution on the previous slide:

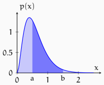
$$[1, 2, 2, 3, 3, 3, 4, 4, 5, 7, 11]$$



Probability density function (PDF)

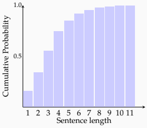
- Continuous variables have *probability density functions*
- $p(x)$ is not a probability (note the notation: we use lowercase p for PDF)
- Area under $p(x)$ sums to 1
- $P(X = x) = 0$
- Non zero probabilities are possible for ranges:

$$P(a \leqslant x \leqslant b) = \int_a^b p(x)\,dx$$



Cumulative distribution function

- $F_X(x) = P(X \leqslant x)$



Length	Prob.	C. Prob.
1	0.16	0.16
2	0.18	0.34
3	0.21	0.55
4	0.19	0.74
5	0.10	0.85
6	0.07	0.91
7	0.04	0.95
8	0.02	0.97
9	0.01	0.99
10	0.01	0.99
11	0.00	1.00

Expected value

- Expected value (mean) of a random variable X is,

$$E[X] = \mu = \sum_{i=1}^{n} P(x_i)x_i = P(x_1)x_1 + P(x_2)x_2 + \ldots + P(x_n)x_n$$

- More generally, expected value of a function of X is

$$E[f(X)] = \sum_x P(x)f(x)$$

- Expected value is a measure of central tendency
- Note: it is not the 'most likely' value
- Expected value is linear

$$E[aX + bY] = aE[X] + bE[Y]$$

Variance and standard deviation

- Variance of a random variable X is,

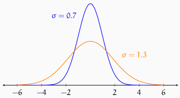
$$\mathrm{Var}(X) = \sigma^2 = \sum_{i=1}^{n} P(x_i)(x_i - \mu)^2 = E[X^2] - (E[X])^2$$

- It is a measure of spread, divergence from the central tendency
- The square root of variance is called standard deviation

$$\sigma = \sqrt{\left(\sum_{i=1}^{n} P(x_i)x_i^2\right) - \mu^2}$$

- Standard deviation is in the same units as the values of the random variable
- Variance is not linear: $\sigma^2_{X+Y} \neq \sigma^2_X + \sigma^2_Y$ (neither the σ)

Example: two distributions with different variances



Short divergence: Chebyshev's inequality

For any probability distribution, and $k > 1$,

$$P(|x - \mu| > k\sigma) \leqslant \frac{1}{k^2}$$

Distance from μ	2σ	3σ	5σ	10σ	100σ
Probability	0.25	0.11	0.04	0.01	0.0001

This also shows why standardizing values of random variables,

$$z = \frac{x - \mu}{\sigma}$$

makes sense (the normalized quantity is often called the z-score).

Median and mode of a random variable

Median is the mid-point of a distribution. Median of a random variable is defined as the number m that satisfies

$$P(X \leqslant m) \geqslant \frac{1}{2} \quad \text{and} \quad P(X \geqslant m) \geqslant \frac{1}{2}$$

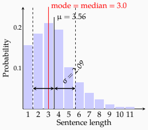
- Median of $1, 4, 5, 8, 10$ is 5
- Median of $1, 4, 5, 7, 8, 10$ is 6

Mode is the value that occurs most often in the data.

- Modes appear as peaks in probability mass (or density) functions
- Mode of $1, 4, 4, 8, 10$ is 4
- Modes of $1, 4, 4, 8, 9, 9$ are 4 and 9

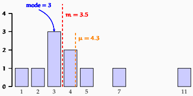
Mode, median, mean, standard deviation

Visualization on sentence length example

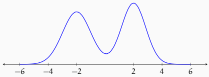


Mode, median, mean

sensitivity to extreme values



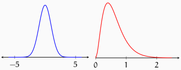
Multimodal distributions



- A distribution is multimodal if it has multiple modes
- Multimodal distributions often indicate confounding variables

Skew

- Another important property of a probability distribution is its *skew*
- symmetric distributions have no skew
- positively skewed distributions have a long *tail* on the right
- negatively skewed distributions have a long left tail

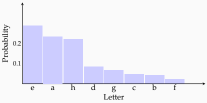


Another example distribution

A probability distribution over letters

- An alphabet with 8 letters and their probabilities of occurrance;

Lett.	a	b	c	d	e	f	g	h
Prob.	0.23	0.04	0.05	0.08	0.29	0.02	0.07	0.22



Probability distributions

- A distribution on a finite set of outcomes can be defined by a vector (or table) of probabilities
- Some random variables (approximately) follow a distribution that can be parametrized with a number of parameters
- For example, Gaussian (or normal) distribution is conventionally parametrized by its mean (μ) and variance (σ^2)
- Common notation we use for indicating that a variable X follows a particular distribution is

$$X \sim \text{Normal}(\mu, \sigma^2) \quad \text{or} \quad X \sim \mathcal{N}(\mu, \sigma^2).$$

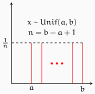
- For the rest of this lecture, we will revise some of the important probability distributions

Probability distributions (cont)

- A probability distribution is called *univariate* if it was defined on scalars
- *multivariate* probability distributions are defined on vectors
- Probability distributions are abstract mathematical objects (functions that map events/outcomes to probabilities)
- A probability distribution is a generative device: it can generate samples
- In most problems, we only have access to a *samples*
- Learning (or *inference*) is often cast as finding an (approximate) distribution from a sample

Uniform distribution (discrete)

- A uniform distribution assigns equal probabilities to all values in range $[a, b]$, where a and b are the parameters of the distribution
- Probabilities of the values outside range is 0
- $\mu = \frac{b+a}{2}$
- $\sigma_2 = \frac{(b-a+1)^2-1}{12}$
- There is also an analogous continuous uniform distribution



Bernoulli distribution

Bernoulli distribution characterizes simple random experiments with two outcomes

- Coin flip: heads or tails
- Spam detection: spam or not
- Predicting gender: female or male

We denote (arbitrarily) one of the possible values with 1 (often called a success), the other with 0 (often called a failure)

$$
\begin{aligned}
P(X = 1) &= p \\
P(X = 0) &= 1 - p \\
P(X = k) &= p^k (1 - p)^{1-k} \\
\mu_X &= p \\
\sigma^2_X &= p(1 - p)
\end{aligned}
$$

Binomial distribution

Binomial distribution is a generalization of Bernoulli distribution to n trials, the value of the random variable is the number of 'successes' in the experiment

$$
\begin{aligned}
P(X = k) &= \binom{n}{k} p^k (1 - p)^{n-k} \\
\mu_X &= np \\
\sigma^2_X &= np(1 - p)
\end{aligned}
$$

Remember that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Categorical distribution

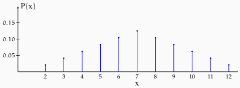
- Extension of Bernoulli to k mutually exclusive outcomes
- For any k-way event, the probability distribution is parametrized by k parameters $p_1, \ldots, p_k$ ($k - 1$ independent parameters) where

$$
\begin{aligned}
\sum_{i=1}^{k} p_i &= 1 \\
E[x_i] &= p_i \\
Var(x_i) &= p_i(1 - p_i)
\end{aligned}
$$

- Similar to Bernoulli–binomial generalization, *multinomial* distribution is the generalization of categorical distribution to n trials

Categorical distribution example

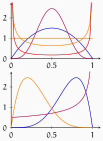
sum of the outcomes from roll of two fair dice



Beta distribution

- Beta distribution is defined in range $[0, 1]$
- It is characterized by two parameters α and β

$$
p(x) = \frac{x^{\alpha-1}(1 - x)^{\beta-1}}{\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}}
$$



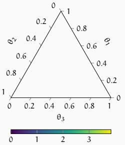
Beta distribution

where do we use it

- A common use is the random variables whose values are probabilities
- Particularly important in Bayesian methods as a conjugate prior of Bernoulli and Binomial distributions
- The *Dirichlet distribution* generalizes Beta distribution to k-dimensional vectors whose components are in range $(0, 1)$ and $\|x\|_1 = 1$.
- Dirichlet distribution is used often in NLP, e.g., *latent Dirichlet allocation* is a well know method for topic modeling

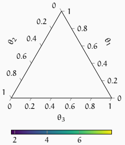
Example Dirichlet distributions

$\theta = (2, 2, 2)$



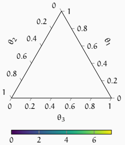
Example Dirichlet distributions

$\theta = (0.8, 0.8, 0.8)$



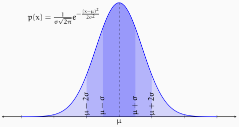
Example Dirichlet distributions

$\theta = (2, 2, 4)$



Gaussian (normal) distribution

$$
p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}
$$



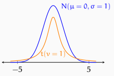
Short detour: central limit theorem

Central limit theorem states that the sum of a large number of independent and identically distributed variables (i.i.d.) is normally distributed.

- Expected value (average) of means of samples from any distribution will be distributed normally
- Many (inference) methods in statistics and machine learning work because of this fact

Student's t-distribution

- T-distribution is another important distribution
- It is similar to normal distribution, but it has heavier tails
- It has one parameter: *degree of freedom* (ν)



Joint and marginal probability

Two or more random variables form a *joint probability distribution*.

An example with letter bigrams:

	a	**b**	**c**	**d**	**e**	**f**	**g**	**h**	
a	0.04	0.02	0.02	0.03	0.05	0.01	0.02	0.06	0.23
b	0.01	0.00	0.00	0.00	0.01	0.00	0.00	0.01	0.04
c	0.02	0.00	0.00	0.00	0.01	0.00	0.00	0.01	0.05
d	0.02	0.00	0.00	0.01	0.02	0.00	0.01	0.02	0.08
e	0.06	0.02	0.01	0.03	0.08	0.01	0.01	0.07	0.29
f	0.00	0.00	0.00	0.00	0.01	0.00	0.00	0.01	0.02
g	0.01	0.00	0.00	0.01	0.02	0.00	0.01	0.02	0.07
h	0.08	0.00	0.00	0.01	0.10	0.00	0.01	0.02	0.22
	0.23	0.04	0.05	0.08	0.29	0.02	0.07	0.22	

Expected values of joint distributions

$$E[f(X,Y)] = \sum_x \sum_y P(x,y) f(x,y)$$

$$\mu_X = E[X] = \sum_x \sum_y P(x,y)x$$

$$\mu_Y = E[Y] = \sum_x \sum_y P(x,y)y$$

We can simplify the notation by vector notation, for $\boldsymbol{\mu} = (\mu_x, \mu_y)$,

$$\boldsymbol{\mu} = \sum_{\mathbf{x} \in XY} \mathbf{x} P(\mathbf{x})$$

where vector $\mathbf{x}$ ranges over all possible combinations of the values of random variables X and Y.

Variances of joint distributions

$$\sigma_X^2 = \sum_x \sum_y P(x,y)(x - \mu_X)^2$$

$$\sigma_Y^2 = \sum_x \sum_y P(x,y)(y - \mu_Y)^2$$

$$\sigma_{XY} = \sum_x \sum_y P(x,y)(x - \mu_X)(y - \mu_Y)$$

- The last quantity is called *covariance* which indicates whether the two variables vary together or not

Again, using vector/matrix notation we can define the *covariance matrix* ($\boldsymbol{\Sigma}$) as

$$\boldsymbol{\Sigma} = E[(\mathbf{x} - \boldsymbol{\mu})^2]$$

Covariance and the covariance matrix

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{YX} & \sigma_Y^2 \end{bmatrix}$$

- The main diagonal of the covariance matrix contains the variances of the individual variables
- Non-diagonal entries are the covariances of the corresponding variables
- Covariance matrix is symmetric ($\sigma_{XY} = \sigma_{YX}$)
- For a joint distribution of k variables we have a covariance matrix of size k × k

Correlation

Correlation is a normalized version of covariance

$$r = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

Correlation coefficient (r) takes values between -1 and 1

1 Perfect positive correlation.

$(0, 1)$ positive correlation: x increases as y increases.

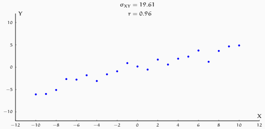
0 No correlation, variables are independent.

$(-1, 0)$ negative correlation: x decreases as y increases.

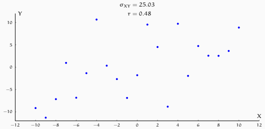
-1 Perfect negative correlation.

Note: like covariance, correlation is a symmetric measure.

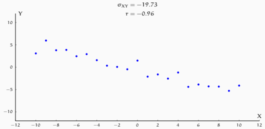
Correlation: visualization (1)



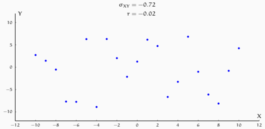
Correlation: visualization (2)



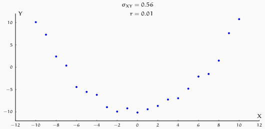
Correlation: visualization (3)



Correlation: visualization (4)



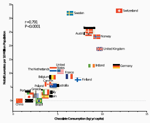
Correlation: visualization (5)



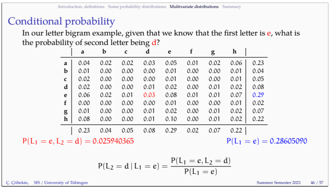
Correlation and independence

- Statistical (in)dependence is an important concept (in ML)
- The correlation (or covariance) of independent random variables is 0
- The reverse is not true: 0 correlation does not imply independence
- Correlation measures a linear dependence (relationship) between two variables, a non-linear dependence is not measured by correlation

Short divergence: correlation and causation



From Messerli (2012).



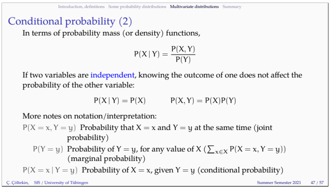
Conditional probability

In our letter bigram example, given that we know that the first letter is e, what is the probability of second letter being d?

	a	b	c	d	e	f	g	h	
a	0.04	0.02	0.02	0.03	0.05	0.01	0.02	0.06	0.23
b	0.01	0.00	0.00	0.00	0.01	0.00	0.00	0.01	0.04
c	0.02	0.00	0.00	0.00	0.01	0.00	0.00	0.01	0.05
d	0.02	0.00	0.00	0.01	0.02	0.00	0.01	0.02	0.08
e	0.06	0.02	0.01	0.03	0.08	0.01	0.01	0.07	0.29
f	0.00	0.00	0.00	0.00	0.01	0.00	0.00	0.01	0.02
g	0.01	0.00	0.00	0.01	0.02	0.00	0.01	0.02	0.07
h	0.08	0.00	0.00	0.01	0.10	0.00	0.01	0.02	0.22
	0.23	0.04	0.05	0.08	0.29	0.02	0.07	0.22	

$P(L_1 = e, L_2 = d) = 0.025940365$ $P(L_1 = e) = 0.28605090$

$$P(L_2 = d \mid L_1 = e) = \frac{P(L_1 = e, L_2 = d)}{P(L_1 = e)}$$



Conditional probability (2)

In terms of probability mass (or density) functions,

$$P(X \mid Y) = \frac{P(X, Y)}{P(Y)}$$

If two variables are independent, knowing the outcome of one does not affect the probability of the other variable:

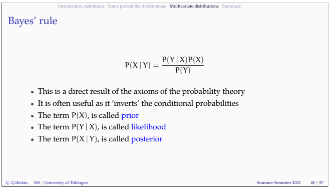
$$P(X \mid Y) = P(X) \qquad P(X, Y) = P(X)P(Y)$$

More notes on notation/interpretation:

$P(X = x, Y = y)$ Probability that $X = x$ and $Y = y$ at the same time (joint probability)

$P(Y = y)$ Probability of $Y = y$, for any value of X ($\sum_{x \in X} P(X = x, Y = y)$) (marginal probability)

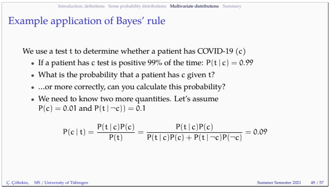
$P(X = x \mid Y = y)$ Probability of $X = x$, given $Y = y$ (conditional probability)



Bayes' rule

$$P(X \mid Y) = \frac{P(Y \mid X)P(X)}{P(Y)}$$

- This is a direct result of the axioms of the probability theory
- It is often useful as it 'inverts' the conditional probabilities
- The term $P(X)$, is called prior
- The term $P(Y \mid X)$, is called likelihood
- The term $P(X \mid Y)$, is called posterior

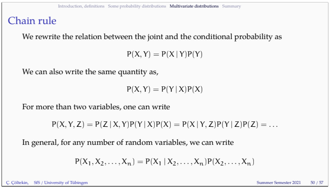


Example application of Bayes' rule

We use a test t to determine whether a patient has COVID-19 (c)

- If a patient has c test is positive 99% of the time: $P(t \mid c) = 0.99$
- What is the probability that a patient has c given t?
- ...or more correctly, can you calculate this probability?
- We need to know two more quantities. Let's assume $P(c) = 0.01$ and $P(t \mid \neg c)) = 0.1$

$$P(c \mid t) = \frac{P(t \mid c)P(c)}{P(t)} = \frac{P(t \mid c)P(c)}{P(t \mid c)P(c) + P(t \mid \neg c)P(\neg c)} = 0.09$$



Chain rule

We rewrite the relation between the joint and the conditional probability as

$$P(X, Y) = P(X \mid Y)P(Y)$$

We can also write the same quantity as,

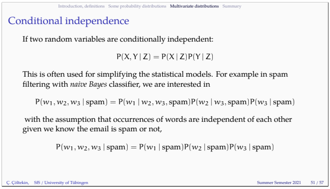
$$P(X, Y) = P(Y \mid X)P(X)$$

For more than two variables, one can write

$$P(X, Y, Z) = P(Z \mid X, Y)P(Y \mid X)P(X) = P(X \mid Y, Z)P(Y \mid Z)P(Z) = \ldots$$

In general, for any number of random variables, we can write

$$P(X_1, X_2, \ldots, X_n) = P(X_1 \mid X_2, \ldots, X_n)P(X_2, \ldots, X_n)$$



Conditional independence

If two random variables are conditionally independent:

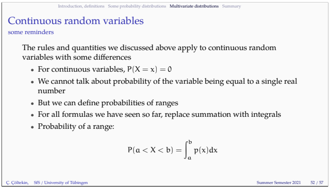
$$P(X, Y \mid Z) = P(X \mid Z)P(Y \mid Z)$$

This is often used for simplifying the statistical models. For example in spam filtering with *naive Bayes* classifier, we are interested in

$$P(w_1, w_2, w_3 \mid \text{spam}) = P(w_1 \mid w_2, w_3, \text{spam})P(w_2 \mid w_3, \text{spam})P(w_3 \mid \text{spam})$$

with the assumption that occurrences of words are independent of each other given we know the email is spam or not,

$$P(w_1, w_2, w_3 \mid \text{spam}) = P(w_1 \mid \text{spam})P(w_2 \mid \text{spam})P(w_3 \mid \text{spam})$$



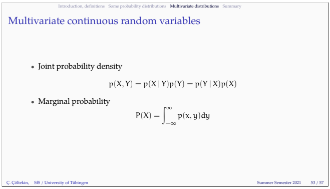
Continuous random variables

some reminders

The rules and quantities we discussed above apply to continuous random variables with some differences

- For continuous variables, $P(X = x) = 0$
- We cannot talk about probability of the variable being equal to a single real number
- But we can define probabilities of ranges
- For all formulas we have seen so far, replace summation with integrals
- Probability of a range:

$$P(a < X < b) = \int_a^b p(x)dx$$



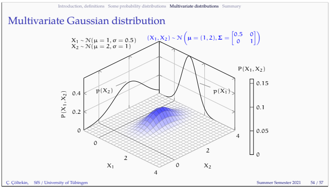
Multivariate continuous random variables

- Joint probability density

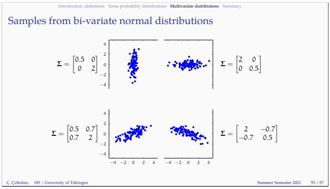
$$p(X, Y) = p(X \mid Y)p(Y) = p(Y \mid X)p(X)$$

- Marginal probability

$$P(X) = \int_{-\infty}^{\infty} p(x, y)dy$$



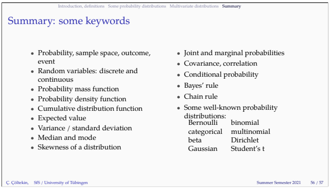
Multivariate Gaussian distribution



Samples from bi-variate normal distributions

$$\Sigma = \begin{bmatrix} 0.5 & 0 \\ 0 & 2 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix}$$

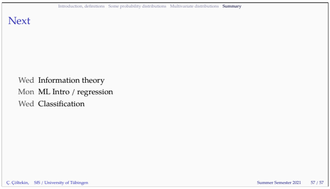
$$\Sigma = \begin{bmatrix} 0.5 & 0.7 \\ 0.7 & 2 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} 2 & -0.7 \\ -0.7 & 0.5 \end{bmatrix}$$



Summary: some keywords

- Probability, sample space, outcome, event
- Random variables: discrete and continuous
- Probability mass function
- Probability density function
- Cumulative distribution function
- Expected value
- Variance / standard deviation
- Median and mode
- Skewness of a distribution
- Joint and marginal probabilities
- Covariance, correlation
- Conditional probability
- Bayes' rule
- Chain rule
- Some well-known probability distributions:

Bernoulli	binomial
categorical	multinomial
beta	Dirichlet
Gaussian	Student's t



Next

Wed	Information theory
Mon	ML Intro / regression
Wed	Classification

References and further reading

- MacKay (2003) covers most of the topics discussed in a way quite relevant to machine learning. The complete book is available freely online (see the link below)
- See Grinstead and Snell (2012) a more conventional introduction to probability theory. This book is also freely available
- For an influential, but not quite conventional approach, see Jaynes (2007)

Chomsky, Noam (1968). “Quine’s empirical assumptions”. In: *Synthese* 19.1, pp. 53–68. DOI: `10.1007/BF00568049`.

Grinstead, Charles Miller and James Laurie Snell (2012). *Introduction to probability*. American Mathematical Society. ISBN: 9780821894149. URL: `http://www.dartmouth.edu/~chance/teaching_aids/books_articles/probability_book/book.html`.

Jaynes, Edwin T (2007). *Probability Theory: The Logic of Science*. Ed. by G. Larry Bretthorst. Cambridge University Press. ISBN: 978-05-2159-271-0.

MacKay, David J. C. (2003). *Information Theory, Inference and Learning Algorithms*. Cambridge University Press. ISBN: 978-05-2164-298-9. URL: `http://www.inference.phy.cam.ac.uk/itprnn/book.html`.

Messerli, Franz H (2012). “Chocolate consumption, cognitive function, and Nobel laureates”. In: *The New England journal of medicine* 367.16, pp. 1562–1564.